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Applied Mathematics Letters 17 (2004) 111–121

Applied Mathematics Letters

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Continuous Wavelet Transforms on the Space $L^2(\mathbf{R}, \mathbb{H}; dx)$

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(Received July 2001; revised and accepted May 2002)

Abstract—Let **P** be the affine group of the real line **R**, and let $\mathbb{H}$ be the set of all quaternions. Thus, $L^2(\mathbf{R}, \mathbb{H}; dx)$ denotes the space of all square integrable $\mathbb{H}$-valued functions. From the viewpoint of square integrable group representations, we study the theory of continuous wavelet transforms on $L^2(\mathbf{R}, \mathbb{H}; dx)$ associated with the group **P**, and give the Calderón reproducing formula.

Keywords—Admissibility condition, Continuous wavelet transform, Calderón reproducing formula.

1. INTRODUCTION

The continuous wavelet transforms are deeply related to the concepts of square integrable group representations (see [1]). First, we recall some basic concepts. Let G be a locally compact group with the left Haar measure $d\mu_l(g)$, let $g \mapsto U(g)(g \in G)$ be an irreducible continuous unitary representation of G in a Hilbert space H. If a nonzero vector $\phi \in H$ satisfies the following admissibility condition:

$$C_\phi = \frac{1}{\|\phi\|_H^2} \int_G \left|\langle \phi, U(g)\phi \rangle_H\right|^2 \, d\mu_l(g) < +\infty, \tag{1}$$

then we say that ϕ is an admissible vector. Write the set of all such vectors by AW. If AW $\neq \emptyset$, then U is called square integrable, and $f \mapsto \langle f, U(g)\phi \rangle_H$ is called the continuous wavelet transform. The usual continuous wavelet transform on real line $\mathbf{R}$ is derived from a natural unitary representation on $L^2(\mathbf{R})$ of the affine group "$\rho x' + x$". Let G be the affine group

This work was originally done at the Department of Mathematics, Nanjing Normal University, Nanjing 210097, P.R. China.
*Supported by the Foundation of the National Natural Science of China (Grant 10071039) and the Foundation of Education Commission of Jiangsu Province.
The authors would like to thank the anonymous referee for apt suggestions for improvement of this paper.

0893-9659/04/$ - see front matter Typeset by $\mathcal{AMS}$-TEX
doi: 10.1016/S0893-9659(03)00228-3

$\mathbf{P} = \{(x,\rho) : x \in \mathbf{R}, \rho > 0\}$, and $H = L^2(\mathbf{R})$, $\mathbf{P}$ has a natural unitary representation U on the space $L^2(\mathbf{R})$ defined by

$$U(x,\rho) f(x') = \frac{1}{\sqrt{\rho}} f\left(\frac{x'-x}{\rho}\right), \qquad f \in L^2(\mathbf{R}). \tag{2}$$

The Hardy space

$$H_+^2(\mathbf{R}) = \left\{ f \in L^2(\mathbf{R}) : \hat{f}(\xi) = 0, \text{ if } \xi \notin \mathbf{R}^+ \right\} \tag{3}$$

and the conjugate Hardy space

$$H_-^2(\mathbf{R}) = \left\{ f \in L^2(\mathbf{R}) : \hat{f}(\xi) = 0, \text{ if } \xi \notin \mathbf{R}^- \right\} \tag{4}$$

are two irreducible subspaces. The restrictions of U on these subspaces are square integrable. If we identify $\mathbf{P}$ with the upper half-plane U^+, then the wavelet transform $W_\phi f(x,\rho) = (1/\sqrt{\rho}) \int_{\mathbf{R}} f(x')\bar{\phi}(x'-x/\rho)\,dx'$ gives an isometric operator from $H_+^2(\mathbf{R})$ (or $H_-^2(\mathbf{R})$) to $L^2(U^+, dx\,d\rho/\rho^2)$, where $f, \phi \in H_+^2(\mathbf{R})$ (or $H_-^2(\mathbf{R})$), and ϕ satisfies the admissibility condition:

$$0 < C_\phi = \int_{\mathbf{R}} \frac{\left|\hat{\phi}(\xi)\right|^2 d\xi}{|\xi|} < +\infty. \tag{5}$$

The wavelet analysis on $L^2(\mathbf{R})$ associated with the group $\mathbf{P}$ has substantial content (see [2,3]), and those results have been extended to higher dimension in different ways (see [4–6]).

In the early part of the nineteenth century, Irish mathematician Hamilton invented the quaternions, since then a lot of authors investigated the properties of the noncommutative associative algebra: the algebra of quaternions $\mathbb{H}$ (see [7,8]). Moreover, some problems of quaternion-valued functions were studied, for a survey of that we refer the reader to [9–12]. Using the complex inner product on $\mathbb{H}$, He [13] established the theory of wavelet analysis on $L^2(\mathbf{C}, \mathbb{H};\, dz)$. Qian [12] introduced the natural quaternion inner product on $\mathbb{H}$. He studied the problems of singular integrals in the quaternion-valued function space, and obtained some interesting results. Therefore, the principal goal of this paper is to study the theory of continuous wavelet transforms on the space $L^2(\mathbf{R}, \mathbb{H};\, dx)$ connected with the quaternion inner product on $\mathbb{H}$. The results in this paper are different from those in [13]. Recently, Xia and Suter [14] and Walden and Serroukh [15] discussed the problems of matrix-valued wavelet analysis. At the end of this section, we will see that the theory of continuous wavelet transforms on $L^2(\mathbf{R}, \mathbb{H};\, dx)$ can be regarded as that on a kind of matrix-valued function space.

This paper is arranged into four sections. The remainder of this section contains preliminaries. In Section 2, we decompose $L^2(\mathbf{R}, \mathbb{H};\, dx)$ into the direct sum of the irreducible invariant subspaces. Section 3 gives the characterization of the admissibility condition. Section 4 obtains the Parseval formula and the Calderón reproducing formula for the wavelet transforms.

Let $\mathbb{H}$ denote the set of all quaternions, i.e.,

$$\begin{aligned} \mathbb{H} &= \mathbf{R} \oplus i\mathbf{R} \oplus j\mathbf{R} \oplus k\mathbf{R} \\ &= \{q = a + ib + jc + kd : a,b,c,d \in \mathbf{R}\}, \end{aligned} \tag{6}$$

where i, j, k satisfy

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j, \qquad i^2 = j^2 = k^2 = -1.$$

For $q \in \mathbb{H}$, we can write q in the form $q = (a+ib) + j(c-id)$. Without loss of generality, we set $q = u + jv$, where $u, v \in \mathbb{C}$. Let q^c denote the conjugate of q, then $q^c = \bar{u} - jv$, where $\bar{u}$ is the complex conjugate of u. Let $q_1 = u_1 + jv_1, q_2 = u_2 + jv_2 \in \mathbb{H}$. Obviously, $(q^c)^c = q, (q_1q_2)^c = q_2^c q_1^c$. We now introduce a mapping $\langle , \rangle_{\mathbb{H}}$ from $\mathbb{H} \times \mathbb{H}$ to $\mathbb{H}$ as follows:

$$\begin{aligned} \langle , \rangle_{\mathbb{H}} : \qquad & \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H} \\ & (q_1, q_2) \longmapsto \langle q_1, q_2 \rangle_{\mathbb{H}} = q_1 q_2^c \\ & \qquad\qquad\qquad = (u_1\bar{u}_2 + \bar{v}_1 v_2) + j(v_1\bar{u}_2 - \bar{u}_1 v_2). \end{aligned} \tag{7}$$

It is easy to verify the following properties.

(1) $\langle q_1, q_2 \rangle_{\mathbb{H}} = \langle q_2, q_1 \rangle_{\mathbb{H}}{}^c$.

(2) For $\alpha, \beta \in \mathbb{H}$, we have

$$\begin{aligned} \langle \alpha q_1 + \beta q_2, q \rangle_{\mathbb{H}} &= \alpha \langle q_1, q \rangle_{\mathbb{H}} + \beta \langle q_2, q \rangle_{\mathbb{H}}, \\ \langle q, \alpha q_1 + \beta q_2 \rangle_{\mathbb{H}} &= \langle q, q_1 \rangle_{\mathbb{H}} \alpha^c + \langle q, q_2 \rangle_{\mathbb{H}} \beta^c. \end{aligned}$$

In particular, if $\alpha, \beta \in \mathbf{R}$, then

$$\langle q, \alpha q_1 + \beta q_2 \rangle_{\mathbb{H}} = \alpha \langle q, q_1 \rangle_{\mathbb{H}} + \beta \langle q, q_2 \rangle_{\mathbb{H}}.$$

(3) $\langle q, q \rangle_{\mathbb{H}} \geq 0$. And $\langle q, q \rangle_{\mathbb{H}} = 0$ if and only if $q = 0$.

In this paper, we adopt $\langle , \rangle_{\mathbb{H}}$ as the inner product on $\mathbb{H}$, and the norm of an arbitrary $q \in \mathbb{H}$ is denoted by $\|q\|_{\mathbb{H}}$. Consider that a function $\mathbf{F}$ is a mapping or transformation from $\mathbf{R}$ to $\mathbb{H}$. Thus, $\mathbf{F}$ can be expressed by $\mathbf{F}(x) = f_1(x) + jf_2(x)$, where f_1, f_2 are all complex-valued functions. The space $L^2(\mathbf{R}, \mathbb{H}; dx)$ consists of all measurable $\mathbb{H}$-valued functions for which the norm of $\mathbf{F}$,

$$\|\mathbf{F}\|_{L^2(\mathbf{R},\mathbb{H};\, dx)} = \left(\int_{\mathbf{R}} \left(|f_1(x)|^2 + |f_2(x)|^2 \right) dx \right)^{1/2},$$

is finite. Let $\mathbf{F}(x) = f_1(x) + jf_2(x), \mathbf{G}(x) = g_1(x) + jg_2(x) \in L^2(\mathbf{R}, \mathbb{H}; dx)$. We define

$$\begin{aligned} \langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} &= \int_{\mathbf{R}} \langle \mathbf{F}, \mathbf{G} \rangle_{\mathbb{H}}\, dx \\ &= \int_{\mathbf{R}} \left[\left(f_1(x)\bar{g}_1(x) + \bar{f}_2(x) g_2(x) \right) + j \left(f_2(x)\bar{g}_1(x) - \bar{f}_1(x) g_2(x) \right) \right] dx. \end{aligned} \tag{8}$$

Thus, $L^2(\mathbf{R}, \mathbb{H}; dx)$ equipped with the above inner product is a Hilbert space. Let

$$L^2(\mathbf{R}, \mathbb{C}^{n\times n}) = \left\{ \beta(x) = \begin{pmatrix} \beta_{11}(x) & \beta_{12}(x) & \cdots & \beta_{1n}(x) \\ \beta_{21}(x) & \beta_{22}(x) & \cdots & \beta_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1}(x) & \beta_{n2}(x) & \cdots & \beta_{nn}(x) \end{pmatrix} : \beta_{kl}(x) \in L^2(\mathbf{R}),\ k, l = 1, 2, \ldots, n \right\},$$

and the complex duplex matrix-valued function space (see [7])

$$L^{2,*}(\mathbf{R}, \mathbb{C}^{2\times 2}) = \left\{ \beta(x) = \begin{pmatrix} \beta_{11}(x) & \beta_{12}(x) \\ \beta_{21}(x) & \beta_{22}(x) \end{pmatrix} : \bar{\beta}_{11}(x) = \beta_{22}(x), -\bar{\beta}_{12}(x) = \beta_{21}(x) \right\}.$$

The theory of wavelet analysis on $L^2(\mathbf{R}, \mathbb{C}^{n\times n})$ was studied in [14,15]. Now we define a mapping Q_m from $L^2(\mathbf{R}, \mathbb{H}; dx)$ to $L^{2,*}(\mathbf{R}, \mathbb{C}^{2\times 2})$ by

$$Q_m : \mathbf{F}(x) = f_1(x) + jf_2(x) \mapsto \begin{pmatrix} f_1(x) & -\bar{f}_2(x) \\ f_2(x) & \bar{f}_1(x) \end{pmatrix} \in L^{2,*}(\mathbf{R}, \mathbb{C}^{2\times 2}). \tag{9}$$

It is easy to verify that the mapping Q_m is a bijection from $L^2(\mathbf{R}, \mathbb{H};\, dx)$ to $L^{2,*}(\mathbf{R}, \mathbb{C}^{2\times 2})$. For any $\mathbf{F}, \mathbf{G} \in L^2(\mathbf{R}, \mathbb{H};\, dx)$, from [14], we have

$$\langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} \mapsto \langle Q_m(\mathbf{F}), Q_m(\mathbf{G}) \rangle_{L^{2,*}(\mathbf{R},\mathbb{C}^{2\times 2})}, \tag{10}$$

$$\|\mathbf{F}\|_{L^2(\mathbf{R},\mathbb{H};\, dx)} = \frac{1}{\sqrt{2}} \|Q_m(\mathbf{F})\|_{L^{2,*}(\mathbf{R},\mathbb{C}^{2\times 2})}. \tag{11}$$

Thus, $L^2(\mathbf{R}, \mathbb{H};\, dx)$ can be regarded as a subspace of $L^2(\mathbf{R}, \mathbb{C}^{2\times 2})$. Let $\mathbf{P}$ be the affine group of the real line $\mathbf{R}$. Then, we know

$$\mathbf{P} = \left\{ (x, \rho) : x \in \mathbf{R}, \rho \in \mathbf{R}^+ \right\}.$$

And $d\mu_l(x, \rho) = dx\, d\rho/\rho^2$ is the left Haar measure of $\mathbf{P}$. The unitary representation U of $\mathbf{P}$ on $L^2(\mathbf{R}, \mathbb{H};\, dx)$ is defined by

$$U(x, \rho)\, \mathbf{F}(x') = \frac{1}{\sqrt{\rho}} \mathbf{F}\left(\frac{x' - x}{\rho}\right). \tag{12}$$

In the following, we shall investigate the wavelet theory on $L^2(\mathbf{R}, \mathbb{H};\, dx)$ associated with the square integrable representations U of $\mathbf{P}$.

2. DECOMPOSITION OF $L^2(\mathbf{R}, \mathbb{H};\, dx)$

Let $u(x)$ be an integrable complex-valued function, whose Fourier transform is the function $\hat{u}(\xi)$ defined by letting

$$\hat{u}(\xi) = \int_{\mathbf{R}} u(x)\, e^{-2i\pi x \xi}\, dx,$$

for all $\xi \in \mathbf{R}$. The Fourier transform of a square integrable function $u(x)$ has a natural definition. Letting $\mathbf{F} \in L^2(\mathbf{R}, \mathbb{H};\, dx)$, we define the Fourier transform of $\mathbf{F}$ by

$$\hat{\mathbf{F}}(\xi) = \hat{f}_1(\xi) + j \hat{f}_2(\xi).$$

Thus, we have the following.

LEMMA 1. *Let $\mathbf{F}, \mathbf{G} \in L^2(\mathbf{R}, \mathbb{H};\, dx)$. Then,*

$$\langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} = \left\langle \hat{\mathbf{F}}, \hat{\mathbf{G}} \right\rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)}. \tag{13}$$

PROOF. A simple computation gives this formula. In fact,

$$\begin{aligned}
\langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R};\mathbb{H};\, dx)} &= \int_{\mathbf{R}} \langle \mathbf{F}, \mathbf{G} \rangle_{\mathbb{H}}\, dx \\
&= \int_{\mathbf{R}} \left[\left(f_1(x)\, \bar{g}_1(x) + \bar{f}_2(x)\, g_2(x) \right) + j \left(f_2(x)\, \bar{g}_1(x) - \bar{f}_1(x)\, g_2(x) \right) \right] dx \\
&= \int_{\mathbf{R}} \left[\left(\hat{f}_1(\xi)\, \bar{\hat{g}}_1(\xi) + \bar{\hat{f}}_2(\xi)\, \hat{g}_2(\xi) \right) + j \left(\hat{f}_2(\xi)\, \bar{\hat{g}}_1(\xi) - \bar{\hat{f}}_1(\xi)\, \hat{g}_2(\xi) \right) \right] d\xi \\
&= \int_{\mathbf{R}} \left\langle \hat{\mathbf{F}}, \hat{\mathbf{G}} \right\rangle_{\mathbb{H}} d\xi \\
&= \left\langle \hat{\mathbf{F}}, \hat{\mathbf{G}} \right\rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)}.
\end{aligned}$$

∎

Let $\mathbf{F} * \mathbf{G}$ denote the convolution of $\mathbf{F}$ and $\mathbf{G}$ defined by

$$(\mathbf{F} * \mathbf{G})(x) = \int_{\mathbf{R}} \mathbf{F}(x - x')\, \mathbf{G}(x')\, dx'.$$

It can be directly verified that

$$\widehat{(\mathbf{F}*\mathbf{G})}(\xi) = \left[\hat{f}_1(\xi)\,\hat{g}_1(\xi) - \hat{\bar{f}}_2(\xi)\,\hat{g}_2(\xi)\right] + j\left[\hat{f}_2(\xi)\,\hat{g}_1(\xi) + \hat{\bar{f}}_1(\xi)\,\hat{g}_2(\xi)\right],$$
$$\hat{\mathbf{F}}(\xi)\,\hat{\mathbf{G}}(\xi) = \left[\hat{f}_1(\xi)\,\hat{g}_1(\xi) - \bar{\hat{f}}_2(\xi)\,\hat{g}_2(\xi)\right] + j\left[\hat{f}_2(\xi)\,\hat{g}_1(\xi) + \bar{\hat{f}}_1(\xi)\,\hat{g}_2(\xi)\right].$$

Thus, $\widehat{(\mathbf{F}*\mathbf{G})}(\xi) \neq \hat{\mathbf{F}}(\xi)\hat{\mathbf{G}}(\xi)$. Let $f_1(x) = \alpha(x) + i\beta(x), f_2(x) = \gamma(x) + i\theta(x)$. If $\bar{\hat{f}}_1(\xi) = \hat{\bar{f}}_1(\xi)$, $\bar{\hat{f}}_2(\xi) = \hat{\bar{f}}_2(\xi)$, namely,

$$\int_{\mathbf{R}} \alpha(x)\sin 2\pi\xi x\,dx = \int_{\mathbf{R}} \beta(x)\sin 2\pi\xi x\,dx = 0$$

and

$$\int_{\mathbf{R}} \gamma(x)\sin 2\pi\xi x\,dx = \int_{\mathbf{R}} \theta(\xi)\sin 2\pi\xi x\,dx = 0,$$

then $\widehat{(\mathbf{F}*\mathbf{G})}(\xi) = \hat{\mathbf{F}}(\xi)\hat{\mathbf{G}}(\xi)$. Let

$$H^{(+,+)} = \left\{\mathbf{F} = f_1(x) + jf_2(x) : f_1, f_2 \in H^2_+(\mathbf{R})\right\}, \tag{14}$$
$$H^{(-,-)} = \left\{\mathbf{F} = f_1(x) + jf_2(x) : f_1, f_2 \in H^2_-(\mathbf{R})\right\}, \tag{15}$$
$$H^{(+,-)} = \left\{\mathbf{F} = f_1(x) + jf_2(x) : f_1 \in H^2_+(\mathbf{R}),\ f_2 \in H^2_-(\mathbf{R})\right\}, \tag{16}$$

and

$$H^{(-,+)} = \left\{\mathbf{F} = f_1(x) + jf_2(x) : f_1 \in H^2_-(\mathbf{R}),\ f_2 \in H^2_+(\mathbf{R})\right\}. \tag{17}$$

Since

$$U\widehat{(x,\rho)}\,\mathbf{F}(\xi) = \rho^{1/2}\left(\hat{f}_1(\rho\xi)\,e^{-2i\pi\xi x} + j\hat{f}_2(\rho\xi)\,e^{-2i\pi\xi x}\right),$$

it is easy to see that $H^{(+,+)}$, $H^{(-,-)}$, $H^{(+,-)}$, and $H^{(-,+)}$ are invariant closed subspaces of $L^2(\mathbf{R},\mathbb{H};\,dx)$ under the unitary representation U of $\mathbf{P}$. Moreover, we can obtain the following.

Lemma 2. *Let $\sigma_i = +$ or $-(i = 1,2)$. Then, $H^{(\sigma_1,\sigma_2)}$ is an irreducible invariant closed subspace of $L^2(\mathbf{R},\mathbb{H};\,dx)$ under the unitary representation U of $\mathbf{P}$.*

Proof. Let W be a nonzero invariant subspace of $H^{(\sigma_1,\sigma_2)}$ under U, $W^\perp$ the orthogonal complement of W in $H^{(\sigma_1,\sigma_2)}$. Taking a function $\mathbf{G} \in W$, $\mathbf{G} \neq 0$. Suppose $\mathbf{F} \in W^\perp$, then we have

$$\langle \mathbf{F}, U(x,\rho)\,\mathbf{G}\rangle_{L^2(\mathbf{R},\mathbb{H};\,dx)} = 0.$$

Notice that

$$\langle \mathbf{F}, U(x,\rho)\,\mathbf{G}\rangle_{L^2(\mathbf{R},\mathbb{H};\,dx)} = \left(\langle f_1, U(x,\rho)\,g_1\rangle_{L^2(\mathbf{R})} + \langle U(x,\rho)\,g_2, f_2\rangle_{L^2(\mathbf{R})}\right)$$
$$+\,j\left(\langle f_2, U(x,\rho)\,g_1\rangle_{L^2(\mathbf{R})} - \langle U(x,\rho)\,g_2, f_1\rangle_{L^2(\mathbf{R})}\right).$$

Taking Fourier transforms with respect to the first variable x on both sides of the above equality, and noting that $\hat{\bar{f}}(\xi) = \bar{\hat{f}}(-\xi)$, we can see that, for all $\rho \in \mathbf{R}^+$,

$$\begin{aligned}&\widehat{\langle \mathbf{F}, U(x,\rho)\,\mathbf{G}\rangle}_{L^2(\mathbf{R},\mathbb{H};\,dx)}(\xi)\\ &= \sqrt{\rho}\left(\hat{f}_1(\xi)\,\bar{\hat{g}}_1(\rho\xi) + \bar{\hat{f}}_2(-\xi)\,\hat{g}_2(-\rho\xi)\right) + j\left(\hat{f}_2(\xi)\,\bar{\hat{g}}_1(\rho\xi) - \bar{\hat{f}}_1(-\xi)\,\hat{g}_2(-\rho\xi)\right) = 0\end{aligned} \tag{18}$$

holds for almost all $\xi \in \mathbf{R}$. We only need to show that $\mathbf{F} = 0$, i.e., $\hat{f}_1(\xi) = 0$, $\hat{f}_2(\xi) = 0$ for almost every $\xi \in \mathbf{R}$.

CASE 1. Let $\mathbf{F} = f_1 + jf_2$, $\mathbf{G} = g_1 + jg_2 \in H^{(+,+)}$. Then, $\mathrm{supp}(f_i), \mathrm{supp}(g_i) \subset [0, +\infty)$ $(i = 1, 2)$. Assume the contrary, let A be a positive measure subset in $[0, +\infty)$, such that $\hat{f}_1(\xi) \neq 0$ for all $\xi \in A$. Since $\mathbf{G} \neq 0$, we know that there exists a positive measure subset B in $[0, +\infty)$, such that $\hat{g}_1(\xi) \neq 0$ or $\hat{g}_2(\xi) \neq 0$ for all $\xi \in B$. Assume that $\hat{g}_1(\xi) \neq 0$ for all $\xi \in B$. Choosing $\rho_0 > 0$ appropriately such that $D = A \cap \rho_0^{-1} B$ is a positive measure set. Thus, $\hat{f}_1(\xi) \neq 0$, $\hat{g}_1(\rho_0\xi) \neq 0$, for all $\xi \in D$, which implies that

$$\hat{f}_1(\xi)\,\bar{\hat{g}}_1(\rho_0\xi) \neq 0, \tag{19}$$

for all $\xi \in D$. Clearly, $D \subset [0, +\infty)$. From (18), we obtain $\hat{f}_1(\xi)\bar{\hat{g}}_1(\rho_0\xi) + \bar{\hat{f}}_2(-\xi)\hat{g}_2(-\rho_0\xi) = 0$ for all $\xi \in \mathbf{R}$. Note that the supports of f_2 and g_2 are all subsets of $[0, +\infty)$, we can deduce that $\hat{f}_1(\xi)\bar{\hat{g}}_1(\rho_0\xi) + \bar{\hat{f}}_2(-\xi)\hat{g}_2(-\rho_0\xi) = \hat{f}_1(\xi)\bar{\hat{g}}_1(\rho_0\xi) = 0$ for all $\xi \in [0, +\infty)$, which contradicts (19). Similarly, if $\hat{g}_2(\xi) \neq 0$ for all $\xi \in B$, then there exists a positive measure subset D^* in $(-\infty, 0]$, such that $\bar{\hat{f}}_1(-\xi)\hat{g}_2(-\rho_0\xi) \neq 0$ for all $\xi \in D^*$. Contradicting the fact that

$$0 = \hat{f}_2(\xi)\,\bar{\hat{g}}_1(\rho_0\xi) - \bar{\hat{f}}_1(-\xi)\,\hat{g}_2(-\rho_0\xi) = \bar{\hat{f}}_1(-\xi)\,\hat{g}_2(-\rho_0\xi),$$

for all $\xi \in (-\infty, 0]$.

CASE 2. Let $\mathbf{F} = f_1 + jf_2, \mathbf{G} = g_1 + jg_2 \in H^{(+,-)}$. Then, $\mathrm{supp}(f_1), \mathrm{supp}(g_1) \subset [0, +\infty)$, $\mathrm{supp}(f_2), \mathrm{supp}(g_2) \subset (-\infty, 0]$. Since $\mathbf{G} \neq 0$, we obviously have $(\bar{\hat{g}}_1(\xi), -\hat{g}_2(\xi)) \neq 0$, which implies that $(\bar{\hat{g}}_1(\xi), -\hat{g}_2(-\xi)) \neq 0$. But, from (18), we can easily find that for all $\rho \in \mathbf{R}^+$,

$$\begin{pmatrix} \hat{f}_1(\xi) & -\bar{\hat{f}}_2(-\xi) \\ \hat{f}_2(\xi) & \bar{\hat{f}}_1(-\xi) \end{pmatrix} \begin{pmatrix} \bar{\hat{g}}_1(\rho\xi) \\ -\hat{g}_2(-\rho\xi) \end{pmatrix} = 0, \tag{20}$$

for almost every $\xi \in \mathbf{R}$. We claim that the matrix $\begin{pmatrix} \hat{f}_1(\xi) & -\bar{\hat{f}}_2(-\xi) \\ \hat{f}_2(\xi) & \bar{\hat{f}}_1(-\xi) \end{pmatrix}$ is degenerate. If this were not the case, then there would exist a positive measure subset D in $[0, +\infty)$ and $\rho_0 > 0$, such that $\begin{pmatrix} \hat{f}_1(\xi) & -\bar{\hat{f}}_2(-\xi) \\ \hat{f}_2(\xi) & \bar{\hat{f}}_1(-\xi) \end{pmatrix}$ is nondegenerate, and $(\bar{\hat{g}}_1(\rho_0\xi), -\hat{g}_2(-\rho_0\xi)) \neq 0$ for all $\xi \in D$. Thus,

$$\begin{pmatrix} \hat{f}_1(\xi) & -\bar{\hat{f}}_2(-\xi) \\ \hat{f}_2(\xi) & \bar{\hat{f}}_1(-\xi) \end{pmatrix} \begin{pmatrix} \bar{\hat{g}}_1(\rho_0\xi) \\ -\hat{g}_2(-\rho_0\xi) \end{pmatrix} \neq 0,$$

for all $\xi \in D$, which contradicts (20). Thus, $(\hat{f}_1(\xi), -\bar{\hat{f}}_2(-\xi)) = \alpha(\hat{f}_2(\xi), \bar{\hat{f}}_1(-\xi))$. Consequently, $\hat{f}_1(\xi) = \hat{f}_2(\xi) = 0$ since $\mathrm{supp}(f_1) \subset [0, +\infty), \mathrm{supp}(f_2) \subset (-\infty, 0]$. The other cases can be proved by the analogous argument.

From the above discussion, we can see that $\mathbf{F}$ is identically zero. Hence, $H^{(\sigma_1,\sigma_2)}$ is irreducible. Thus, we complete the proof of Lemma 2. ∎

Here we call $H^{(+,+)}$ and $H^{(-,-)}$ the Hardy space and the conjugate Hardy space of $L^2(\mathbf{R}, \mathbb{H}; dx)$, respectively. $H^{(+,-)}$ and $H^{(-,+)}$ are the "mixed" Hardy spaces. From Lemma 2, we can see that $H^{(+,+)}$ and $H^{(-,-)}$ are mutually orthogonal, while $H^{(+,-)}$ and $H^{(-,+)}$ are not so. Thus, we can get the following.

THEOREM 1.

$$L^2(\mathbf{R}, \mathbb{H}; dx) = H^{(+,+)} \bigoplus H^{(-,-)}. \tag{21}$$

3. WAVELET TRANSFORM ON $L^2(\mathbf{R}, \mathbb{H}; dx)$

In order to define the wavelet transform on $L^2(\mathbf{R}, \mathbb{H}; dx)$, we first give the characterization of the admissible condition.

Let $\Phi \in L^2(\mathbf{R}, \mathbb{H}; dx)$, $\Phi \neq 0$, if it satisfies

$$C_\Phi = \frac{1}{\|\Phi\|^2_{L^2(\mathbf{R},\mathbb{H};dx)}} \int_{\mathbf{R}\times\mathbf{R}^+} \left\| \langle \Phi, U(x,\rho)\,\Phi\rangle_{L^2(\mathbf{R},\mathbb{H};dx)} \right\|^2_{\mathbb{H}} \frac{dx\,d\rho}{\rho^2} < +\infty, \tag{22}$$

then we call Φ an admissible wavelet, and write $\Phi \in \mathrm{AW}$. Let $\Phi = \phi_1 + j\phi_2 \in H^{(+,+)}$. By Lemma 1 and (18), we can get

$$\begin{aligned}
&\int_{\mathbf{R}\times\mathbf{R}^+} \left\| \langle \Phi, U(x,\rho)\,\Phi\rangle_{L^2(\mathbf{R},\mathbb{H};dx)} \right\|^2_{\mathbb{H}} \frac{dx\,d\rho}{\rho^2} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} \left\| \left[\hat{\phi}_1(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) + \hat{\phi}_2(-\xi)\,\bar{\hat{\phi}}_2(-\rho\xi)\right] + j\left[\hat{\phi}_2(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) - \hat{\phi}_2(-\rho\xi)\,\bar{\hat{\phi}}_1(-\xi)\right] \right\|^2_{\mathbb{H}} \frac{d\xi\,d\rho}{\rho} \\
&= \left(\int_{\mathbf{R}^+\times\mathbf{R}^+} \left\| \hat{\phi}_1(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) + j\hat{\phi}_2(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) \right\|^2_{\mathbb{H}} \frac{d\xi\,d\rho}{\rho} \right) \\
&\quad + \left(\int_{\mathbf{R}^-\times\mathbf{R}^+} \left\| \hat{\phi}_2(-\xi)\,\bar{\hat{\phi}}_2(-\rho\xi) - j\hat{\phi}_2(-\rho\xi)\,\bar{\hat{\phi}}_1(-\xi) \right\|^2_{\mathbb{H}} \frac{d\xi\,d\rho}{\rho} \right) \\
&= \int_{\mathbf{R}} \left(\left|\hat{\phi}_1(\xi)\right|^2 + \left|\hat{\phi}_2(\xi)\right|^2 \right) \left(\int_{\mathbf{R}^+} \left(\left|\hat{\phi}_1(\rho\xi)\right|^2 + \left|\hat{\phi}_2(\rho\xi)\right|^2 \right) \frac{d\rho}{\rho} \right) d\xi \\
&= \left(\int_{\mathbf{R}^+} \frac{\left\|\hat{\Phi}(\xi)\right\|^2_{\mathbb{H}} d\xi}{\xi} \right) \|\Phi\|^2_{L^2(\mathbf{R},\mathbb{H};dx)}.
\end{aligned}$$

Similarly, if $\Phi \in H^{(-,-)}$, then we can easily obtain

$$\int_{\mathbf{R}\times\mathbf{R}^+} \left\| \langle \Phi, U(x,\rho)\,\Phi\rangle_{L^2(\mathbf{R},\mathbb{H};dx)} \right\|^2_{\mathbb{H}} \frac{dx\,d\rho}{\rho^2} = \left(\int_{\mathbf{R}^-} \frac{\left\|\hat{\Phi}(\xi)\right\|^2_{\mathbb{H}} d\xi}{|\xi|} \right) \|\Phi\|^2_{L^2(\mathbf{R},\mathbb{H};dx)}.$$

Let $\Phi = \phi_1 + j\phi_2 \in H^{(+,-)}$. Noting that $\mathrm{supp}(\hat{\phi}_1) \subset [0,+\infty), \mathrm{supp}(\hat{\phi}_2) \subset (-\infty, 0]$, we have

$$\begin{aligned}
&\int_{\mathbf{R}\times\mathbf{R}^+} \left\| \langle \Phi, U(x,\rho)\,\Phi\rangle_{L^2(\mathbf{R},\mathbb{H};dx)} \right\|^2_{\mathbb{H}} \frac{dx\,d\rho}{\rho^2} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} \left\| \left[\hat{\phi}_1(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) + \hat{\phi}_2(-\xi)\,\bar{\hat{\phi}}_2(-\rho\xi)\right] + j\left[\hat{\phi}_2(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) - \hat{\phi}_2(-\rho\xi)\,\bar{\hat{\phi}}_1(-\xi)\right] \right\|^2_{\mathbb{H}} \frac{d\xi\,d\rho}{\rho} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} \left| \left[\hat{\phi}_1(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) + \hat{\phi}_2(-\xi)\,\bar{\hat{\phi}}_2(-\rho\xi)\right] \right|^2 \frac{d\xi\,d\rho}{\rho} \\
&\le \int_{\mathbf{R}} \left(\left|\hat{\phi}_1(\xi)\right|^2 + \left|\hat{\phi}_2(-\xi)\right|^2 \right) \left(\int_{\mathbf{R}^+} \left(\left|\hat{\phi}_1(\rho\xi)\right|^2 + \left|\hat{\phi}_2(-\rho\xi)\right|^2 \right) \frac{d\rho}{\rho} \right) d\xi \\
&= \left(\int_{\mathbf{R}} \frac{\left\|\hat{\Phi}(\xi)\right\|^2_{\mathbb{H}} d\xi}{|\xi|} \right) \|\Phi\|^2_{L^2(\mathbf{R},\mathbb{H};dx)}.
\end{aligned}$$

Thus, we can see that the restriction of U on $H^{(\sigma_1,\sigma_2)}$ is square integrable.

THEOREM 2. *Let $\Phi \in H^{(+,+)}(\text{or} H^{(-,-)}), \Phi \neq 0$. Then, $\Phi \in \mathrm{AW}$ if and only if*

$$C_\Phi = \int_{\mathbf{R}} \frac{\left\|\hat{\Phi}(\xi)\right\|^2_{\mathbb{H}} d\xi}{|\xi|} < +\infty. \tag{23}$$

Let $\Phi = \phi_1 + j\phi_2, \Psi = \psi_1 + j\psi_2 \in \mathrm{AW}$. The inner product on AW is defined by

$$\langle \Phi, \Psi\rangle_{\mathrm{AW}} = \int_{\mathbf{R}} \left(\bar{\hat{\phi}}_1(\xi)\,\hat{\psi}_1(\xi) + \bar{\hat{\phi}}_2(\xi)\,\hat{\psi}_2(\xi) \right) \frac{d\xi}{|\xi|}.$$

Let $\Phi \in \mathrm{AW}, \mathbf{F} \in L^2(\mathbf{R}, \mathbb{H}; dx)$. We define the continuous wavelet transform with respect to Φ by

$$(W_\Phi \mathbf{F})(x, \rho) = \langle \mathbf{F}, U(x, \rho)\, \Phi \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} \,. \tag{24}$$

In the sequel, we just state the wavelet transforms on $H^{(+,+)}$. The case on $H^{(-,-)}$ can be dealt with similarly. Let U^+ be the upper half-plane, i.e.,

$$U^+ = \{(x,y) : x \in \mathbf{R},\ y > 0\}.$$

Then, $\mathbf{P}$ can be identified with U^+. Let $\Phi, \Psi \in \mathrm{AW} \cap H^{(+,+)}, \mathbf{F}, \mathbf{G} \in H^{(+,+)}$. Naturally, we expect that the Parseval formula of the wavelet transform

$$\int_{\mathbf{R}\times\mathbf{R}^+} (W_\Phi \mathbf{F})(x,\rho)\,(W_\Psi \mathbf{G})^c(x,\rho)\, \frac{dx\, d\rho}{\rho^2} = \langle \Phi, \Psi \rangle_{\mathrm{AW}} \langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} \tag{25}$$

holds. Unfortunately, this is not the case. We now give an example to illustrate this fact.

Let $\Phi = \phi_1$, $\Psi = \psi_1 \in \mathrm{AW} \cap H^{(+,+)}$, $\mathbf{F} = f_1 + j f_2$, $\mathbf{G} = g_1 + j g_2 \in H^{(+,+)}$. Then,

$$\begin{aligned}
&\langle W_\Phi \mathbf{F}, W_\Psi \mathbf{G} \rangle_{L^2(\mathbf{U}^+,\mathbb{H};\, dx\, d\rho/\rho^2)} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} (W_\Phi \mathbf{F})(x,\rho)\,(W_\Psi \mathbf{G})^c(x,\rho)\, \frac{dx\, d\rho}{\rho^2} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} \left\{ \left[\hat{f}_1(\xi)\, \bar{\hat{\phi}}_1(\rho\xi) + j \hat{f}_2(\xi)\, \bar{\hat{\phi}}_1(\rho\xi) \right] \left[\bar{\hat{g}}_1(\xi)\, \hat{\psi}_1(\rho\xi) - j \hat{g}_2(\xi)\, \bar{\hat{\psi}}_1(\rho\xi) \right] \right\} \frac{d\xi\, d\rho}{\rho} \\
&= \int_{\mathbf{R}^+\times\mathbf{R}^+} \left\{ \left[\bar{\hat{\phi}}_1(\rho\xi)\, \hat{\psi}_1(\rho\xi)\, \hat{f}_1(\xi)\, \bar{\hat{g}}_1(\xi) + \hat{\phi}_1(\rho\xi)\, \bar{\hat{\psi}}_1(\rho\xi)\, \bar{\hat{f}}_2(\xi)\, \hat{g}_2(\xi) \right] \right. \\
&\quad \left. + j \left[\bar{\hat{\phi}}_1(\rho\xi)\, \hat{\psi}_1(\rho\xi)\, \hat{f}_2(\xi)\, \bar{\hat{g}}_1(\xi) - \hat{\phi}_1(\rho\xi)\, \bar{\hat{\psi}}_1(\rho\xi)\, \bar{\hat{f}}_1(\xi)\, \hat{g}_2(\xi) \right] \right\} \frac{d\xi\, d\rho}{\rho}.
\end{aligned}$$

Generally,

$$\langle W_\Phi \mathbf{F}, W_\Psi \mathbf{G} \rangle_{L^2(\mathbf{U}^+,\mathbb{H};\, dx\, d\rho/\rho^2)} \neq \langle \Phi, \Psi \rangle_{\mathrm{AW}} \langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} \,.$$

But if $\bar{\hat{\phi}}_1(\xi)\hat{\psi}_1(\xi)$ is a real function for all $\xi \in \mathbf{R}$, then one can obtain the Parseval formula. In Section 4, we shall give the condition of Parseval formula.

4. REPRODUCING FORMULA

THEOREM 3.

(1) *Let* $\Phi = \phi_1 + j\phi_2$, $\Psi = \psi_1 + j\psi_2 \in \mathrm{AW} \cap H^{(+,+)}$, $\mathbf{F} = f_1 + j f_2$, $\mathbf{G} = g_1 + j g_2 \in H^{(+,+)}$, $\bar{\hat{\phi}}_l(\xi)\hat{\psi}_l(\xi)$ $(l = 1, 2)$ *are all real functions. Then,*

$$\langle W_\Phi \mathbf{F}, W_\Psi \mathbf{G} \rangle_{L^2(\mathbf{U}^+,\mathbb{H};\, dx\, d\rho/\rho^2)} = \langle \Phi, \Psi \rangle_{\mathrm{AW}} \langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} \,. \tag{26}$$

(2) *Let* $\Phi = \phi_1 + j\phi_2$, $\Psi = \psi_1 + j\psi_2 \in \mathrm{AW} \cap H^{(+,+)}$, $\mathbf{F}, \mathbf{G} \in H^{(+,+)}$, $\mathbf{F} = f_1$, $\mathbf{G} = g_1$, *or* $\mathbf{G} = j g_2$. *Then,*

$$\langle W_\Phi \mathbf{F}, W_\Psi \mathbf{G} \rangle_{L^2(\mathbf{U}^+,\mathbb{H};\, dx\, d\rho/\rho^2)} = \langle \Phi, \Psi \rangle_{\mathrm{AW}} \langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} \,. \tag{27}$$

(3) *Let* $\Phi = \phi_1 + j\phi_2$, $\Psi = \psi_1 + j\psi_2 \in \mathrm{AW} \cap H^{(+,+)}$, $\mathbf{F}, \mathbf{G} \in H^{(+,+)}$, $\mathbf{F} = j f_2$, $\mathbf{G} = j g_2$, *or* $\mathbf{G} = g_1$. *Then,*

$$\langle W_\Phi \mathbf{F}, W_\Psi \mathbf{G} \rangle_{L^2(\mathbf{U}^+,\mathbb{H};\, dx\, d\rho/\rho^2)} = \langle \Psi, \Phi \rangle_{\mathrm{AW}} \langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R},\mathbb{H};\, dx)} \,. \tag{28}$$

PROOF.

(1) Let $\Phi = \phi_1 + j\phi_2$, $\Psi = \psi_1 + j\psi_2 \in \mathrm{AW} \cap H^{(+,+)}$, $\mathbf{F} = f_1 + jf_2$, $\mathbf{G} = g_1 + jg_2 \in H^{(+,+)}$, $\bar{\hat{\phi}}_l(\xi)\hat{\psi}_l(\xi)$ $(l = 1,2)$ are all real functions. Note that for any $q \in \mathbb{H}$, $\alpha \in \mathbb{C}$, $\beta \in \mathbf{R}$, $j\alpha = \bar{\alpha}j$, $\beta q = q\beta$. By Lemma 1 and (18), we therefore obtain

$$
\begin{aligned}
&\langle W_\Phi \mathbf{F}, W_\Psi \mathbf{G} \rangle_{L^2(\mathbf{U}^+, \mathbb{H};\, dx\, d\rho/\rho^2)} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} (W_\Phi \mathbf{F})(x,\rho)\, (W_\Psi \mathbf{G})^c(x,\rho)\, \frac{dx\, d\rho}{\rho^2} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} \left\{ \left[\hat{f}_1(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) + \bar{\hat{f}}_2(-\xi)\,\hat{\phi}_2(-\rho\xi) \right] + j \left[\hat{f}_2(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) - \bar{\hat{f}}_1(-\xi)\,\hat{\phi}_2(-\rho\xi) \right] \right\} \\
&\quad \times \left\{ \left[\bar{\hat{g}}_1(\xi)\,\hat{\psi}_1(\rho\xi) + \hat{g}_2(-\xi)\,\bar{\hat{\psi}}_2(-\rho\xi) \right] - j \left[\hat{g}_2(\xi)\,\bar{\hat{\psi}}_1(\rho\xi) - \bar{\hat{g}}_1(-\xi)\,\hat{\psi}_2(-\rho\xi) \right] \right\} \frac{d\xi\, d\rho}{\rho} \\
&= \int_{\mathbf{R}^+\times\mathbf{R}^+} \left\{ \left[\hat{f}_1(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) + j\hat{f}_2(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) \right] \left[\bar{\hat{g}}_1(\xi)\,\hat{\psi}_1(\rho\xi) - j\hat{g}_2(\xi)\,\bar{\hat{\psi}}_1(\rho\xi) \right] \right\} \frac{d\xi\, d\rho}{\rho} \\
&\quad + \int_{\mathbf{R}^-\times\mathbf{R}^+} \left\{ \left[\bar{\hat{f}}_2(-\xi)\,\hat{\phi}_2(-\rho\xi) - j\bar{\hat{f}}_1(-\xi)\,\hat{\phi}_2(-\rho\xi) \right] \right. \\
&\quad \times \left. \left[\hat{g}_2(-\xi)\,\bar{\hat{\psi}}_2(-\rho\xi) + j\bar{\hat{g}}_1(-\xi)\,\hat{\psi}_2(-\rho\xi) \right] \right\} \frac{d\xi\, d\rho}{\rho} \\
&= \int_{\mathbf{R}^+\times\mathbf{R}^+} \left\{ \left[\hat{f}_1(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) + j\hat{f}_2(\xi)\,\bar{\hat{\phi}}_1(\rho\xi) \right] \left[\bar{\hat{g}}_1(\xi)\,\hat{\psi}_1(\rho\xi) - j\hat{g}_2(\xi)\,\bar{\hat{\psi}}_1(\rho\xi) \right] \right. \\
&\quad + \left. \left[\bar{\hat{f}}_2(\xi)\,\hat{\phi}_2(\rho\xi) - j\bar{\hat{f}}_1(\xi)\,\hat{\phi}_2(\rho\xi) \right] \left[\hat{g}_2(\xi)\,\bar{\hat{\psi}}_2(\rho\xi) + j\bar{\hat{g}}_1(\xi)\,\hat{\psi}_2(\rho\xi) \right] \right\} \frac{d\xi\, d\rho}{\rho} \\
&= \int_{\mathbf{R}^+\times\mathbf{R}^+} \left\{ \left[\hat{f}_1(\xi) + j\hat{f}_2(\xi) \right] \left(\bar{\hat{\phi}}_1(\rho\xi)\,\hat{\psi}_1(\rho\xi) \right) \left[\bar{\hat{g}}_1(\xi) - j\hat{g}_2(\xi) \right] \right. \\
&\quad + \left. \left[\bar{\hat{f}}_2(\xi) - j\bar{\hat{f}}_1(\xi) \right] \left(\hat{\phi}_2(\rho\xi)\,\bar{\hat{\psi}}_2(\rho\xi) \right) \left[\hat{g}_2(\xi) + j\bar{\hat{g}}_1(\xi) \right] \right\} \frac{d\xi\, d\rho}{\rho} \\
&= \int_{\mathbf{R}^+\times\mathbf{R}^+} \left\{ \left[\bar{\hat{\phi}}_1(\rho\xi)\,\hat{\psi}_1(\rho\xi) + \hat{\phi}_2(\rho\xi)\,\bar{\hat{\psi}}_2(\rho\xi) \right] \left\langle \hat{\mathbf{F}}, \hat{\mathbf{G}} \right\rangle_{\mathbb{H}} \right\} \frac{d\xi\, d\rho}{\rho} \\
&= \int_{\mathbf{R}^+\times\mathbf{R}^+} \left[\bar{\hat{\phi}}_1(\rho\xi)\,\hat{\psi}_1(\rho\xi) + \bar{\hat{\phi}}_2(\rho\xi)\,\hat{\psi}_2(\rho\xi) \right] \left\langle \hat{\mathbf{F}}, \hat{\mathbf{G}} \right\rangle_{\mathbb{H}} \frac{d\xi\, d\rho}{\rho} \\
&= \langle \Phi, \Psi \rangle_{\mathrm{AW}} \langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R}, \mathbb{H};\, dx)} .
\end{aligned}
$$

Especially, if $\hat{\phi}_l(\xi)$ and $\hat{\psi}_l(\xi)$ $(l = 1,2)$ are real functions, then formula (26) is also valid. Letting $\Phi = \Psi, \mathbf{F} = \mathbf{G}$, we have

$$\int_{\mathbf{R}\times\mathbf{R}^+} \|(W_\Phi \mathbf{F})(x,\rho)\|_{\mathbb{H}}^2 \frac{dx\, d\rho}{\rho^2} = C_\Phi \|\mathbf{F}\|^2_{L^2(\mathbf{R}, \mathbb{H};\, dx)} . \tag{29}$$

From (29), we can see that the wavelet transform W_Φ is also an isometric operator from $H^{(+,+)}$ to $L^2(\mathbf{U}^+, \mathbb{H};\, dx\, d\rho/\rho^2)$ (up to a multiple).

Part (2) can be proved by slightly modifying the argument used in the proof of Part (1).

(3) Let $\Phi = \phi_1 + j\phi_2$, $\Psi = \psi_1 + j\psi_2 \in \mathrm{AW} \cap H^{(+,+)}$, $\mathbf{F} = jf_2$, $\mathbf{G} = jg_2 \in H^{(+,+)}$. We first note that

$$\langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbf{R}, \mathbb{H};\, dx)} = \int_{\mathbf{R}} \bar{\hat{f}}_2(\xi)\hat{g}_2(\xi)\, d\xi .$$

Thus, we can get

$$\begin{aligned}
&\langle W_\Phi \mathbf{F}, W_\Psi \mathbf{G}\rangle_{L^2(\mathbf{U}^+,\mathbb{H};\, dx\, d\rho/\rho^2)} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} (W_\Phi \mathbf{F})(x,\rho)\,(W_\Psi \mathbf{G})^c(x,\rho)\,\frac{dx\,d\rho}{\rho^2} \\
&= \int_{\mathbf{R}\times\mathbf{R}^+} \left\{\left[\bar{\hat{f}}_2(-\xi)\,\hat{\phi}_2(-\rho\xi) + j\hat{f}_2(\xi)\,\bar{\hat{\phi}}_1(\rho\xi)\right]\left[\hat{g}_2(-\xi)\,\bar{\hat{\psi}}_2(-\rho\xi) - j\hat{g}_2(\xi)\,\bar{\hat{\psi}}_1(\rho\xi)\right]\right\}\frac{d\xi\,d\rho}{\rho} \\
&= \int_{\mathbf{R}^+\times\mathbf{R}^+} \left\{\left[\bar{\hat{f}}_2(\xi)\,\hat{\phi}_2(\rho\xi)\,\hat{g}_2(\xi)\,\bar{\hat{\psi}}_2(\rho\xi) + \bar{\hat{f}}_2(\xi)\,\hat{\phi}_1(\rho\xi)\,\hat{g}_2(\xi)\,\bar{\hat{\psi}}_1(\rho\xi)\right]\right\}\frac{d\xi\,d\rho}{\rho} \\
&= \langle\Psi,\Phi\rangle_{\mathrm{AW}}\,\langle\mathbf{F},\mathbf{G}\rangle_{L^2(\mathbf{R},\mathbb{H};\,dx)}\,.
\end{aligned}$$

The other case can be proved in a similar way. ∎

Another important consequence of Theorem 3 is the following Calderón reproducing formula. We state it as a corollary.

COROLLARY 1. *Let $\Phi = \phi_1 + j\phi_2, \in \mathrm{AW}\cap H^{(+,+)}$, $\mathbf{F} = f_1 + jf_2 \in H^{(+,+)}$. Then,*

$$\mathbf{F}(x') = \frac{1}{C_\Phi}\int_{\mathbf{R}\times\mathbf{R}^+} (W_\Phi\mathbf{F})(x,\rho)\,U(x,\rho)\,\Phi(x')\,\frac{dx\,d\rho}{\rho^2}, \tag{30}$$

with convergence of the integral in the weak sense.

We now give an example. Let $\alpha > -1$, $l \in \mathbf{Z}^+$, where $\mathbf{Z}^+ = \{0,1,2,\dots\}$. We define ϕ_l^α in terms of the Fourier transform by

$$\hat{\phi}_l^\alpha(\xi) = \begin{cases} \Gamma(\alpha+1)^{-1/2}\binom{\alpha+l}{l}^{-1/2}(2\xi)^{(\alpha+1)/2}\,e^{-\xi}L_l^{(\alpha)}(2\xi), & \text{if } \xi \ge 0, \\ 0, & \text{if } \xi < 0, \end{cases} \tag{31}$$

where $L_l^{(\alpha)}$ denotes the Laguerre polynomials defined by

$$L_l^{(\alpha)}(\xi) = \frac{1}{l!}e^{\xi}\xi^{-\alpha}\left(\frac{d}{d\xi}\right)^l\left(e^{-\xi}\xi^{l+\alpha}\right).$$

Let $\Phi_{k,l}^\alpha(x) = \phi_k^\alpha(x) + j\phi_l^\alpha(x)$. Obviously, $C_{\Phi_{k,l}^\alpha} = 2$. Thus, for any $\mathbf{F}\in L^2(\mathbf{R},\mathbb{H};\,dx)$, we have

$$\mathbf{F}(x') = \frac{1}{C_{\Phi_{k,l}^\alpha}}\int_{\mathbf{R}\times\mathbf{R}^+}\left(W_{\Phi_{k,l}^\alpha}\mathbf{F}\right)(x,\rho)\,U(x,\rho)\,\Phi_{k,l}^\alpha(x')\,\frac{dx\,d\rho}{\rho^2}. \tag{32}$$

Especially, if we let $\Phi = \phi_k^1(x)$, $\mathbf{F}(x) = f_1(x)$, then the analogous formula as above is just the inversion formula (2.1) for the wavelet transform on $L^2(\mathbf{R})$ in [16].

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